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A REVIEW OF MARCHING PROCEDURES FOR PARABOLIZED NAVIER-STOKES E--ETC(U)
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FOR PARABOLIZED NAVIER-STOKES EQUATIONS

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A REVIEW OF MARCHING PROCEDURES
FOR PARABOLIZED NAVIER-STOKES EQUATIONS

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Abstract

Marching techniques for the parabolized Navier Stokes equations are considered. With the full pressure interaction and prescribed edge pressure these equations are weakly elliptic in subsonic zones. A minimum marching step size (Δx_{min}), proportional to the total thickness (y_M) of the subsonic layer, exists. However, for thin subsonic boundary layers ($y_M << 1$) and with $\Delta x = O(y_M)$, stable and accurate solutions are possible. With forward differencing of the axial pressure gradient the procedure can be made unconditionally stable; a global iteration procedure, requiring only the storage of the pressure term, has been demonstrated for a separated flow problem. Solutions for incompressible boundary layer-like flows, for internal flows, and for supersonic flow over a cone at incidence with a coupled strongly implicit procedure are presented.

1. Introduction

It has now been generally accepted that boundary layer methodology can be extended to the so-called parabolized Navier-Stokes (PNS) equations for a significant variety of flow problems. In a recent paper, Davis and Rubin[1] have reviewed several viscous

flow computations in which parabolized or thin layer techniques have been applied in order to accurately determine the flow characteristics. This publication also reviews some of the early history of the PNS development.

The purpose of the present paper is to discuss some recent investigations using the PNS equations. In particular, we are concerned here with efficient three-dimensional algorithms, a clearer understanding of the limits of applicability of PNS marching techniques, and pressure interaction relaxation for separated flows and other problems where upstream influence is of importance. In this regard, three solution procedures are considered: (1) single-sweep "boundary layer-like" marching for two and three-dimensional flows; (2) multiple sweep iteration or global pressure relaxation, where upstream influence and possibly axial flow separation are important, but regions of subsonic flow are small; and (3) global relaxation where subsonic flow domains are large. For the latter two classes of problems, the analysis draws heavily on that of interacting boundary layers and inviscid subsonic relaxation methods where applicable.

In the course of proceeding to specific examples, a brief review of the limitations associated with PNS marching or relaxation is necessary. The PNS equations, which for simplicity are given here only for "two-dimensional incompressible flow," are as follows:

$$u_t + uu_x + vu_y = - p_x + \frac{1}{R} u_{yy} \quad , \tag{1a}$$

$$v_t + uv_x + vv_y = -p_y + (\frac{1}{R} v_{yy}) \,, \tag{1b}$$

$$u_x + v_y = 0. \tag{1c}$$

R is the Reynolds number; x is denoted the axial flow direction and y the normal direction. The system (1) differs from the complete Navier-Stokes equations only by the omission of the axial diffusion terms. Strictly speaking the inclusion of the v_{yy} term in (1b) is inconsistent with the omission of u_{xx} in (1a). Either the former should be neglected (this is probably a more appropriate definition of the PNS equations) or the latter retained. In fact, these terms have little effect on any of the results presented here. The mathematical character of (1) is controlled by the p_x term in (1a). When p_x is prescribed (assumed known), the system is parabolic. This was the case in the original merged layer analysis of Rudman and Rubin[2], where for hypersonic cold wall flow p_x can, in fact, be neglected in (1a). It should be emphasized, however, that axial pressure gradients are still present and are evaluated through the momentum equation (1b), and the energy equation in the compressible case.

When the p_x term in (1a) is retained implicitly (not prescribed), the system (1) is no longer parabolic, as an elliptic pressure or acoustic interaction occurs in regions of subsonic flow.[1] The "parabolic" form for the velocities has led to the expression parabolized Navier-Stokes equations. This pressure interaction also appears for boundary layer equations, when p_x is not prescribed. The resulting upstream interaction has been

analyzed by Lighthill[3], who demonstrated the existence of exponential growing solutions in the interaction zone. Similar behavior has been encountered with marching procedures for the PNS equations. The primary difference between the boundary layer and PNS equations is that for the latter the pressure interaction is manifested through both the outer pressure boundary formulation and the normal momentum equation (1b).

2. Single Sweep Marching

For problems where upstream influence and axial flow separation are not significant, it is natural to consider the system (1) by boundary layer marching techniques, i.e., backward differences are applied for all x-gradients. If p_x is prescribed, this approach is quite acceptable as the equations are in fact parabolic. For implicit numerical schemes, the marching calculation should be unconditionally stable for all Δx marching steps, see Davis and Rubin[1] for additional references. On the other hand, if p_x is assumed unknown, the "elliptic" pressure interaction of Lighthill is introduced and therefore the exponential growth, representative of upstream influence, can be anticipated. Lubard and Helliwell[4] have examined the stability of the backward difference approximation for p_x in (1a) and they have shown that for $\Delta x < (\Delta x)_{min}$ instability or departure solutions will occur. Similar results were found earlier by Lin and Rubin.[5] For $\Delta x > (\Delta x)_{min}$ the marching scheme is stable. Therefore $(\Delta x)_{min}$ would appear to represent a measure of the upstream elliptic

interaction. For $\Delta x < (\Delta x)_{min}$ the marching scheme attempts tc represent this interaction and therefore the Lighthill behavior should be recovered.

Since the backward difference formula for p_x does not provide any upstream contribution, it does not properly represent the differential form for $\Delta x < (\Delta x)_{min}$. When the backward difference approximation is less representative of p_x, the error introduced serves to reduce $(\Delta x)_{min}$. For example, at $x = x_i$, $(p_{i-1} - p_{i-2})/\Delta x$ is less severe than $(p_i - p_{i-1})/\Delta x$, and for p_x, prescribed, $(\Delta x)_{min} = 0$. In view of this behavior, several investigators, Vigneron et al.[6], Yanenko et al.[7] and Lin and Rubin[8] have attempted to eliminate the pressure interaction by incorporating "small" inconsistencies into the difference approximation. They have assumed (1) a variable Δx in the difference form for p_x such that $\Delta x > (\Delta x)_{min}$ locally[6], (2) "regularization" functions of the type $(1/\sqrt{R})\ f(u_x, v_x, p_x)$ as modified coefficients for the uu_x and p_x terms in (1a)[7], and (3) the use of finite temporal iteration to modify the convection velocity in each step of the marching procedure.[8] Each of these techniques introduces some inconsistency into the difference equations in subsonic zones; reasonable results have been obtained with these methods for certain problems. In order for these techniques to be effective, the inconsistency must be large enough to suppress the elliptic character, yet small enough to maintain an acceptable order of accuracy.

The PNS model has been considered in some detail by Rubin and Lin.[9] For the system (1) with a backward difference for

$\partial/\partial x$ and central y differences, the linear von Neumann stability analysis leads to the following condition for the eigenvalues λ:[9]

$$\frac{u(\lambda-1) + 4b\ \lambda\sin^2\frac{\beta}{2} + I\ c\ \lambda\sin\beta}{u(\lambda-1)\ \cos^2\frac{\beta}{2} + 4b\ \lambda\sin^2\frac{\beta}{2} + I\ c\ \lambda\sin\beta} = \frac{(\lambda-1)\ F(\lambda)}{4a^2\lambda^2\sin^2\frac{\beta}{2}} \tag{2}$$

where $a = \Delta x/\Delta y$, $b = \Delta x/R\Delta y^2$, $c = av$, $I = \sqrt{-1}$, $F = (\lambda-1)$.

It can be shown that the value of λ_{max} is closely related to the highest frequency mode, so that when the number of grid points across the layer $N >> 1$, $\beta \approx \pi/(N-1)$. Equation (2) then takes the simplified form

$$\frac{(\lambda-1)\ F(\lambda)}{A^2\lambda^2} = 1\ , \tag{3}$$

where $A = \pi\Delta x/y_M$ and y_M is the layer thickness; $y_M = (N)\Delta y$. The condition (3) indicates that $\Delta x/y_M$ is the relevant stability parameter. From (3), the marching procedure will be stable for

$$A = \pi\Delta x/y_M > 2\ .$$

Therefore

$$(\Delta x)_{min} \approx y_M\ . \tag{4}$$

The complete numerical solution of (2), for all β, has been obtained, and the analytic result $(\Delta x)_{min} \lesssim \frac{2}{\pi}\,y_M$ is confirmed. The extent of the elliptic numerical interaction is of the order of the thickness of the total layer. If the system (1) is used to solve boundary layer problems, then $y_M = 0(R^{-1/2})$ and

therefore $(\Delta x)_{min} = 0(R^{-1/2})$. For interaction regions where triple deck[11] structure is applicable, $(\Delta x)_{min} = \frac{2}{\pi} y_M = 0(R^{-3/8})$ or the extent of the Lighthill[3] upstream influence. This would tend to confirm the idea of a limited elliptic zone contained in the PNS formulation. For $\Delta x > (\Delta x)_{min}$, this elliptic effect is suppressed. When the upstream influence is negligible, this inconsistency should have little effect on the solution. For truly interactive flows the ellipticity must be retained and the global forward-difference concept discussed in the next section is required.

The results for the PNS and other "transonic" equations clearly indicate that step sizes of the order of the subsonic region, which for supersonic mainstreams is $0(R^{-1/2})$ or $0(Re^{-3/8})$ in a triple deck region, will provide stable and accurate solutions for flows in which upstream effects are not dominant. In a later section, where a strongly implicit algorithm is introduced to obtain marching solutions for the supersonic flow over a cone at incidence, this $(\Delta x)_{min}$ dependency on y_M will be shown for the compressible PNS equations.

3. Multiple Sweep Marching-Global Iteration

If consistency, for $\Delta x \to 0$, of the difference formulation is to be achieved, or if upstream influence is important and/or separation occurs, then backward differencing of the p_x term should be rejected. With any form of forward or central differencing for p_x, relaxation (multiple marching sweeps or global iteration) is required. Three possibilities will be discussed:

(1) forward differencing, (2) central differencing, and (3) use of the Poisson equation for pressure.

Forward differencing for p_x introduces the upstream value p_{i+1}, which is prescribed in each marching sweep, and also has the advantage of including the local pressure value p_i. This provides for coupling of the pressure-velocity system (1) and allows for a free surface pressure interaction. This is important for problems where axial flow separation occurs.

At this point, a few remarks concerning the role of the p_x term for separated flow are relevant. In recent years there has been considerable analysis of free pressure-boundary layer interactions for separated flows.[10] There is general agreement that, for limited separation bubbles, boundary layer equations can be used to calculate such flows. Moreover, the singularity at separation does not appear if the $p'(x)$ term is not prescribed but allowed to develop. Inverse methods, in which the displacement thickness or shear stress is prescribed during the marching step and then updated in subsequent relaxation sweeps have been considered, as have procedures in which temporal terms are retained in order to introduce the pressure or displacement thickness implicitly. In these procedures, the $p'(x)$ term is replaced with an interaction expression (displacement slope for supersonic flow[10] or Cauchy integral for subsonic flow[11]). The local pressure or displacement thickness then appears implicitly and is coupled with the velocity evaluation. In all of these interaction analyses, those components of the p_x approximation

that introduce upstream terms are updated during the relaxation sweeps.

From interacting boundary layer analysis we can then conclude that if the elliptic character is to be modeled consistently, the p_x representation for the PNS equations should introduce downstream contributions. If the separation singularity is to be circumvented in any relaxation sweep, a free pressure interaction through the outer boundary condition or through the y-momentum equation (1b) must be introduced. Forward differencing would appear to satisfy both of these constraints. The stability analysis for equations (1) has been extended in reference 9 for a variety of p_x approximations. For forward differencing of p_x, the stability condition (4) is modified solely by the factor $F(\lambda)$, such that $F(\lambda) = -\lambda$. From (4) it can be inferred that for $\beta = \pi/N-1$, this procedure is unconditionally stable. The stability curves for general β values are given in reference 9. Forward differencing is unconditionally stable for all β at all R. It is significant, however, that as the convective velocity v increases both eigenvalues asymptote to one. So that stability is marginal when the subsonic region is large.

During the first sweep of the global iteration procedure the value p_{i+1} must be prescribed by an initial guess. p_{i+1} can be chosen as a constant equal to the boundary condition at the outer boundary, or the surface, or some combination thereof. Since the variation of p across the subsonic layer is small, any of these values will generally suffice.

In order to test the applicability of forward differencing for p_x two boundary layer problems have been considered with the full PNS system (1):

(i) Flat Plate: $u = p = 1$ at $y = y_M$;

$$u = v = 0 \quad \text{at} \quad y = 0 \; .$$

(ii) Separation Bubble:

$$\left.\begin{array}{l} u = p = 1, \; x < 0 \\ u = 1-x, \; p_x = uu_x, \; 0 \le x \le 0.25 \\ u = 0.75, \; p_x = 0, \; x \ge 0.25 \end{array}\right\} \begin{array}{c} \text{at} \\ y = y_M \end{array}$$

$$u = v = 0 \quad \text{at} \quad y = 0.$$

The pressure gradient p_x is forward differenced. All other x derivatives are backward differenced. These are neglected in regions of reverse flow. All y derivatives are central differenced, except for the continuity equation (1c) and momentum equation (1b) where the trapezoidal rule is used. Multiple sweeps or global iteration was stable and converged for $(\Delta x)_{min} \approx (y_M/6)$; the value of $(\Delta x)_{min} = y_M/60$ was also tested and with forward differencing was stable. It is significant that <u>in this relaxation procedure it is necessary to store only the pressure field for each successive iteration level</u>. The velocities are re-evaluated during each marching sweep. The results are in excellent agreement with published results for both problems. The free surface pressure interaction introduced by the y-moemntum equation has eliminated the separation singularity for the bubble

problem. The value of Δx equal to one-sixth the boundary layer thickness $y_M = 0(R^{-1/2})$ appears to be adequate. For the smaller value of $(\Delta x)_{min} = y_M/60$ or with $y_M = 0(R^{-3/8})$, i.e., the triple deck interaction length, rather than $R^{-1/2}$ the boundary layer length scale, some variations in the solution were obtained. These are not given here. From the stability results, we note that the value of Δx can be made arbitrarily small; however, for the present examples, this is unnecessary. For the cone geometry, to be considered in a following section, considerably smaller values of Δx are used. Some typical results for the boundary layer examples are shown in figures (1) to (4).

Fig. 1

Fig. 2

Fig. 3

Fig. 4

For the flat plate case, comparisons between the PNS and Blasius solution are shown, for $R = 10^3$ and 10^7, in figure (1) for the velocity components, and in figure (2) for the surface skin friction coefficient C_f. The agreement is quite reasonable. The maximum error occurs at the surface and this can be seen from the figures. Additional results for the pressure variation across the boundary layer are given in reference 9. The pressure p_{i+1} is updated during each sweep of the global iteration procedure. For the initial iteration $p_{i+1}(x,y)$ was taken equal to the prescribed edge value; i.e., $p(x_{i+1},y) = p(x_{i+1},y_M)$. During the relaxation process small pressure variations are calculated across the boundary layer. The qualitative agreement with the third-order Blasius pressure distribution is good.[9] Solutions for the separation bubble case are given in figure (3) for typical isovels, and in figure (4) for the surface pressure variation. The predicted separation point value of $x_{sep} = 0.1180$ is close to the boundary layer value of 0.1198. Both separation and reattachment points exhibited smooth transitions and convergence. Since the outer boundary conditions were fixed and the second-order Cauchy integral displacement condition was not imposed, the free interaction was manifested solely through the y-momentum equation (1b). Inclusion of the displacement boundary condition should have a slight effect on the solutions. Convergence of the global relaxation procedure is quite rapid. Only five to ten iterations are required.[9] Of course, for the problems considered here the pressure variation across the layer

is small so that the initial guess is quite good. In view of the stability analysis previously discussed, and since the PNS system includes all of the elements of both boundary layer and triple deck equations, the present solutions with $y_M = O(R^{-3/8})$ should reproduce the results obtained with these approximations. Detailed comparisons will be the subject of future studies.

If central differencing is used for p_x the downstream point p_{i+1} is introduced once again; however, the value p_i no longer appears and the y-momentum equation will be uncoupled from the velocities unless p_i is re-introduced. Several possibilities exist: (1) p_i appears directly through the outer boundary condition, as in interacting boundary layer theory;[10,11] (2) a temporal relaxation term p_τ is introduced in (1a). This has been used in ADI solutions for interacting boundary layers;[10] (3) a K-R[12] approximation, where forward differencing is corrected during the relaxation sweeps, is applied.

From the stability analysis for each sweep of the marching procedure, it is seen that with central differencing, the function $F(\lambda) = -1$ in (2). This is an unconditionally unstable condition and therefore further reinforces the need for a p_i contribution. With an appropriate p_i contribution $F(\lambda) = (\sigma\lambda - 1)$, where σ reflects the p_i term. From (2), the marching procedure is stabilized conditionally for all $\sigma \neq 0$ (recall the earlier $(\Delta x)_{min}$ condition for $\sigma = 1$); however, for $\sigma \leq -1$, unconditional stability results. Actual experience with the various p_x approximations for the compressible PNS system is discussed in a following section on flow over a cone at incidence.

Finally, a stability analysis for convergence of the global iteration procedure has also been considered. When p_x is treated implicitly, to some degree, preliminary conclusions are that convergence is assured. On the other hand, if p_x is treated explicitly, i.e., from a previous marching solution, it would appear that the global iteration procedure will diverge. The pressure results solely from the uncoupled normal momentum equation (1b).

4. Global Iteration for Subsonic Flow

For problems where subsonic regions are not confined to thin layers, single sweep backward differencing for p_x will generally not be reliable since the stability condition requires that $\Delta x \geq (\Delta x)_{min} = O(y_M) = O(1)$. Therefore, global relaxation for the pressure interaction is required in all cases.

Two procedures for calculating such flows have been considered. In the first, the pressure interaction is evaluated with the Poisson form of the pressure equation. In the second, which is currently under development, the pressure is coupled directly to the elliptic velocity solver by a splitting procedure in the spirit of Dodge.[13] There are a number of significant differences however that allow for a completely consistent global relaxation formulation. For both of these formulations a Poisson operator appears explicitly in the equations and therefore the elliptic character of the equations is further strengthened. Only the former approach shall be described here in any detail. Solutions are presented for an asymmetric channel having a moderate

constriction. The complete analysis is given in reference 14. A short review is presented here.

Equations (1a) and (1b) are rewritten in the form:

$$p_x = F_1 - u_t$$

$$p_y = F_2 - v_t$$

so that differentiating and adding we find

$$\nabla^2 p = F_{1x} + F_{2y} - (u_x + v_y)_t \tag{5}$$

The continuity equation (1c) is replaced with (5). In the global iteration procedure the condition (1c) is applied in obtaining appropriate boundary values for p and this equation is satisfied through the convergence of the pressure solver.

The primary differences between this method or the velocity-split approach for subsonic flows, and the global pressure relaxation procedure described previously for thin subsonic regions are two-fold: (1) the pressure gradient in the momentum equations are treated as given from the previous iteration and therefore the pressure calculation is uncoupled from that for the velocities, and (2) the global relaxation includes not only the pressure but the velocities as well. In view of the temporal gradients appearing in (1a), (1b) and (5), all variables are required from previous iterations. Solutions cannot be obtained in the steady state mode ($\Delta t = \infty$) discussed in the previous section. A line relaxation procedure coupling (5) with (1a), (1b) was not tested. The pressure equations were solved

independently with an ADI or SOR technique. The temporal gradients in (1a) and (1b) allow for smooth passage through separated flow regions.

A typical solution is shown in figure (5). These PNS solutions are compared with full Navier-Stokes results in reference 14. The agreement is good, as the axial diffusion effects are quite small even in the regions of separated flow. This provides another example where steady state marching procedures, with global pressure relaxation, can be used effectively to solve the PNS equations or Navier-Stokes equations where the influence of axial diffusion is small. A similar approach for unbounded subsonic regions is being tested with the velocity-split method mentioned previously.

Fig. 5

5. Example: Cone at Incidence - Supersonic Flow

In order to test the global pressure relaxation procedure for the complete compressible PNS equations, the supersonic flow over a sharp cone at incidence has been considered.[15] The pressure gradient p_x in (1a) has been approximated with forward, central and backward (single sweep) differences. Lubard and Helliwell[4] and Lin and Rubin[5,8] among others have already investigated the latter case. Our primary interest in this regard is to evaluate the limiting value of $(\Delta x)_{min}$ in each of the cases and to determine whether the forward difference

approximation retains the effective unconditional stability predicted in the previous analysis.

In a recent study, Schiff and Steger[16] have presented a global iteration method, but not with the intent of treating problems where upstream interaction or axial flow separation are important. An estimate for p_x is obtained by an initial backward sweep with $\Delta x \geq (\Delta x)_{min}$. Subsequently, p_x is treated as known from the previous iterative value of p. In addition, the sublayer approximation presented by Rubin and Lin (see reference 5) is applied in order to reduce $(\Delta x)_{min}$; i.e., p_x is assumed constant across the subsonic portion of the boundary layer. As noted earlier, the convergence analysis for procedures in which p_x is treated explicitly, i.e., from the previous sweep, would indicate that these global relaxation methods are unstable. In reference 16 the appearance of oscillations is noted after four global iterations.

A second important feature of the present analysis, that is described in detail in reference 15, is the application of the coupled strongly implicit procedure of Rubin and Khosla[17] for the cross plane (normal to the axial flow or x direction) solution. This is considered to be an improvement over ADI or SOR or other splitting techniques as there is an immediate coupling of all the boundaries, i.e., shock, body, lee and wind planes. The strongly implicit character of the algorithm also appears desirable for capturing imbedded shock waves and for evaluating secondary flow separation at larger angles of incidence. In addition,

from earlier studies[17] convergence rates are improved, a direct steady-state solution is possible, and artificial dissipation has not been required for iterative convergence.

The compressible PNS equations are given by an expanded form of (1) with appropriate energy and state equations. All x derivatives are backward or K-R[12] differenced except for p_x which is forward or central differenced in certain cases. A sublayer approximation is not assumed. All y derivatives are central differenced. The trapezoidal rule is used for the continuity and y-momentum equations. Boundary layer-like marching is applied in the x-direction; the normal velocity v is prescribed only at the surface. The outer value of v is coupled with and determined by the Rankine-Hugoniot conditions at the shock. This condition also provides for mass conservation in the shock layer.

The strongly implicit algorithm[17] is used to couple the velocities u, v, w for the calculation in the (y,ζ) cross-plane. The temperature is obtained independently from a similar algorithm and the pressure is updated from the normal (y) momentum equation. The strongly implicit algorithm is of the form:

$$\vec{V}_{ij} = \overrightarrow{GM}_{ij} + T_{1_{ij}} \vec{V}_{i-1,j} + T_{2_{ij}} \vec{V}_{i,j+1} \tag{6}$$

where

$$\vec{V}_{ij} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{ij} .$$

The algorithm (6) is scalarized to improve computational efficiency. The coefficients $\overrightarrow{GM}_{ij}$, $(\vec{T}_{1,2})_{ij}$ are determined from recursive formulas whereby

$$\overrightarrow{GM}_{ij} = f(\overrightarrow{GM}_{i+1}, \overrightarrow{GM}_{i,j-1}, \dots) \tag{7}$$

etc.[17] The boundary conditions on all surfaces are imposed either in (6) or (7). Symmetry conditions are used for the wind and lee planes and the Rankine-Hugoniot relations are imposed at the outer shock wave.

Typical solutions are given in figures (6) and (7) for the shock layer thickness and the heat transfer coefficient. Convergence criteria are 10^{-5} for $\vec{V}$ and 10^{-7} for p and T. The difference grid was quite coarse in the cross plane, 41 x 31 points. With finer grids the accuracy can be improved.[15] For $x/\delta(x) \geq 2$, where $\delta(x)$ is the shock layer thickness, the subsonic layer thickness $y_M \leq 0.2\ \delta$. Several numerical experiments were made in order to estimate the values of $(\Delta x)_{min}$. The results are given in Table 1. Converged results were obtained with central differencing for several marching steps; however, the solutions were oscillatory and extremely inaccurate. When these calculations were continued further, the instability predicted by the

Fig. 6

Fig. 7

relation (4) soon leads to divergence of the solution. With the backward and forward differences, behavior similar to that found for the incompressible equation (1) was recovered here. For the backward p_x calculations, $(\Delta x)_{min}$ appears to be approaching an asymptotic limit as R increases. For the incompressible problem, at $R = 10^4$, $(\Delta x)_{min}/y_M \approx 0.22$, as opposed to the value 0.25 obtained for the cone flow. For the forward p_x calculations, the present results confirm those for the incompressible equation; i.e., unconditional stability. Finally, a very weak stability condition on Δx is obtained when the modified "forward-difference" condition is applied, see Table 1. As seen from the solutions presented here, this approximation is quite good for the cone geometry. Solutions for angles of incidence up to 45° are presented in reference 15.

R	10^3	10^4
Backward p_x	0.2	0.25
Modified "Forward" p_x	10^{-3}	2×10^{-4}
Forward p_x	0	0
Central p_x	Unstable for 0(1)	Unstable for 0(1)

Table 1. Estimated Values of $(\Delta x)_{min}/y_M$; $M_\infty = 7.95$; $\alpha = 18°$.

6. Summary

The parabolized Navier-Stokes equations have been considered for subsonic and supersonic flows. It has been shown that with single sweep marching and backward differencing for all axial derivatives, including the pressure, the elliptic influence is numerically suppressed for marching steps $\Delta x \geq (\Delta x)_{min} \approx y_M$ where y_M is the thickness of the subsonic zone. For subsonic boundary layers or the PNS equations with supersonic outer flow conditions, $y_M = 0(R^{-1/2})$ and therefore $(\Delta x)_{min} = 0(R^{-1/2})$. For triple deck regions, $y_M = 0(R^{-3/8})$ or $(\Delta x)_{min} = 0(R^{-3/8})$. For problems with large regions of subsonic flow $(\Delta x)_{min} = 0(1)$ and large truncation errors can be expected.

If global relaxation is considered, i.e., multiple marching sweeps, then central differencing for p_x is unstable, but forward p_x differencing is unconditionally stable. Forward differencing also has the desirable property of allowing for a complete pressure coupling with the velocities and a free surface pressure interaction. Therefore, separation regions can be evaluated with this global relaxation method. Examples for a flat plate boundary layer and for a separation bubble have been presented.

For supersonic outer flows the global relaxation method requires only that the pressure be retained from the previous iteration. Therefore, computer storage is minimal. With large regions of subsonic flow, a global iteration method is also presented; however, pressure and velocity data is required in this procedure and therefore computer storage will be increased.

Solutions are presented for separated channel flow.

Numerical experiments have been conducted for the supersonic flow over a cone at incidence. A coupled strongly implicit numerical algorithm was applied with backward, central and forward differencing for p_x. The solutions confirm the analytic stability results for the incompressible equations. Backward differencing is conditionally stable ($\Delta x \geq (\Delta x)_{min} \approx y_M$); central differencing is unstable, and forward differencing is unconditionally stable. In view of the results obtained herein, we conclude that for flows, with thin subsonic layers, forward differencing for the p_x term leads to an optimal global pressure relaxation procedure, with free pressure interaction and minimum stability limitations. Global relaxation solutions have been obtained for subsonic flows; however, optimization of such techniques requires further study.

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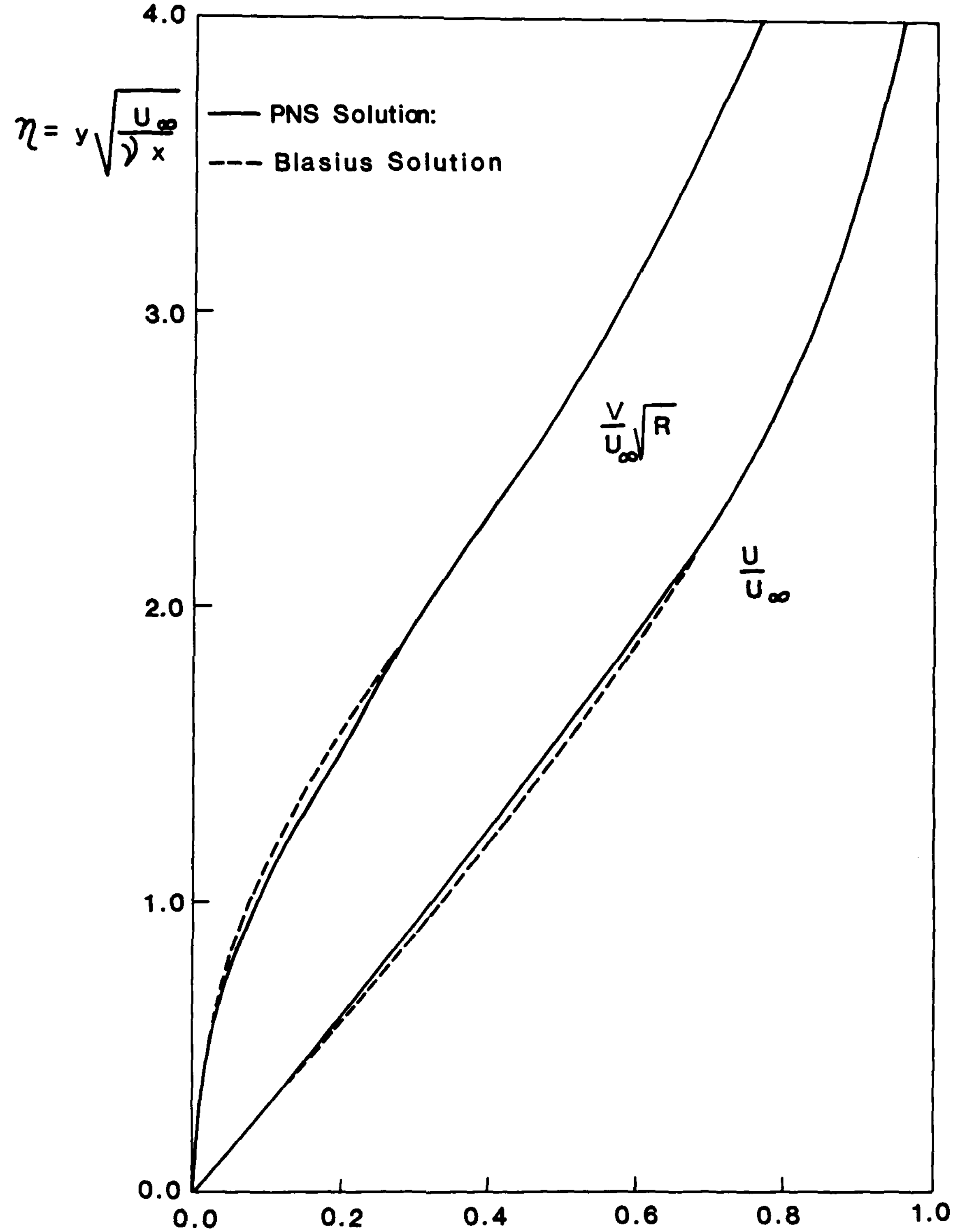

FIG. 1 COMPARISON OF THE PNS VELOCITY PROFILES WITH THE BLASIUS SOLUTIONS: $R = 10^3$, 10^4, $N = 101$

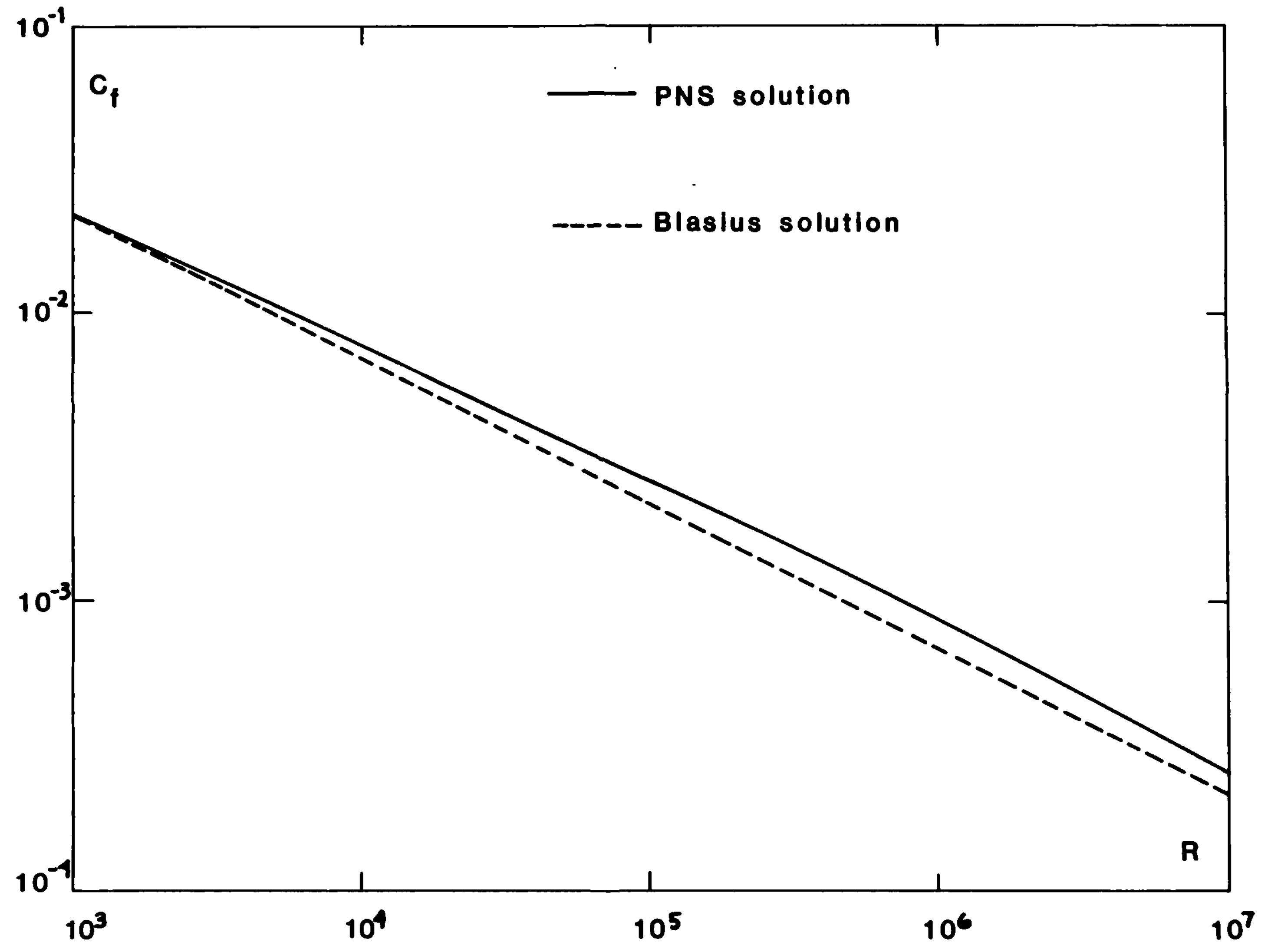

FIG. 2 COMPARISON OF THE LOCAL SKIN FRICTION COEFFICIENT FOR A FLAT PLATE

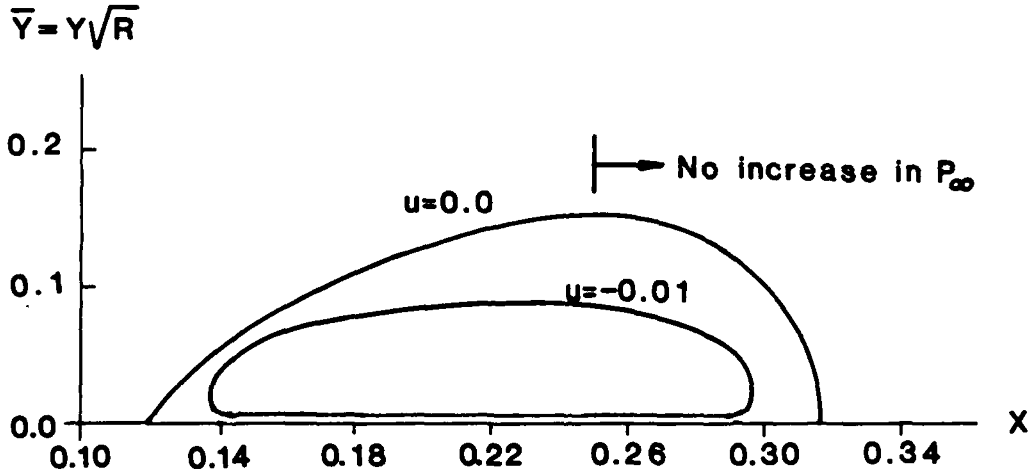

FIG. 3 ISOVELS FOR SEPARATION BUBBLE: PNS EQUATIONS

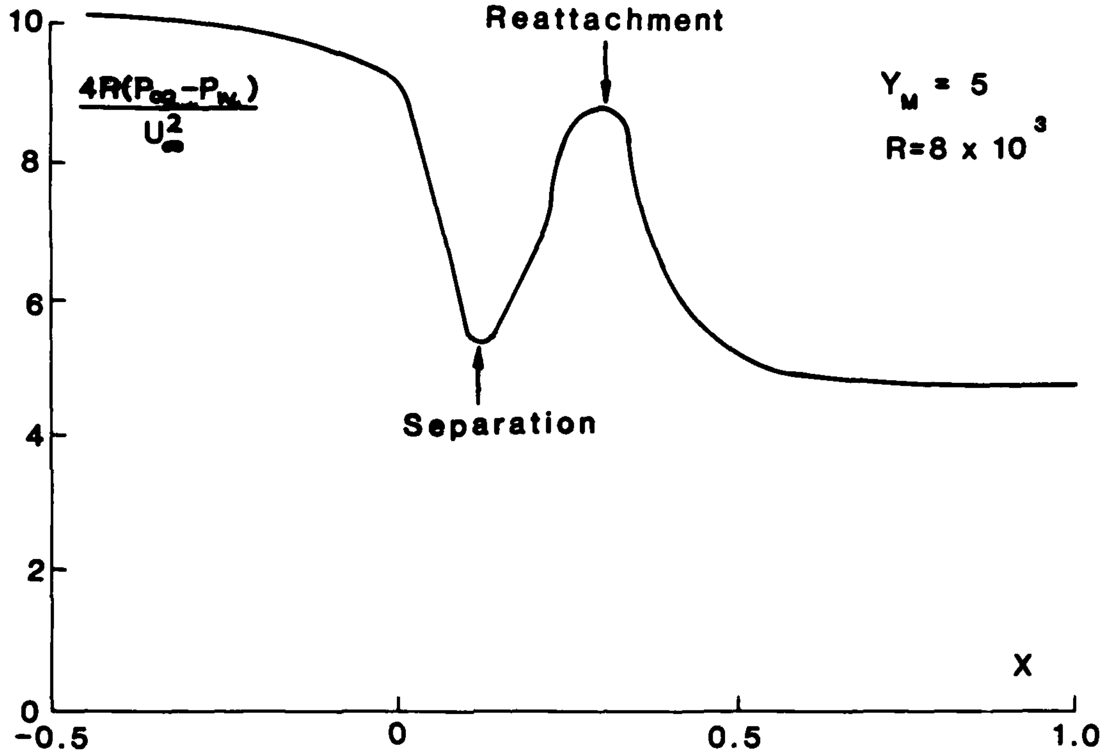

FIG. 4 WALL PRESSURE VARIATION THROUGH THE SEPARATION ZONE

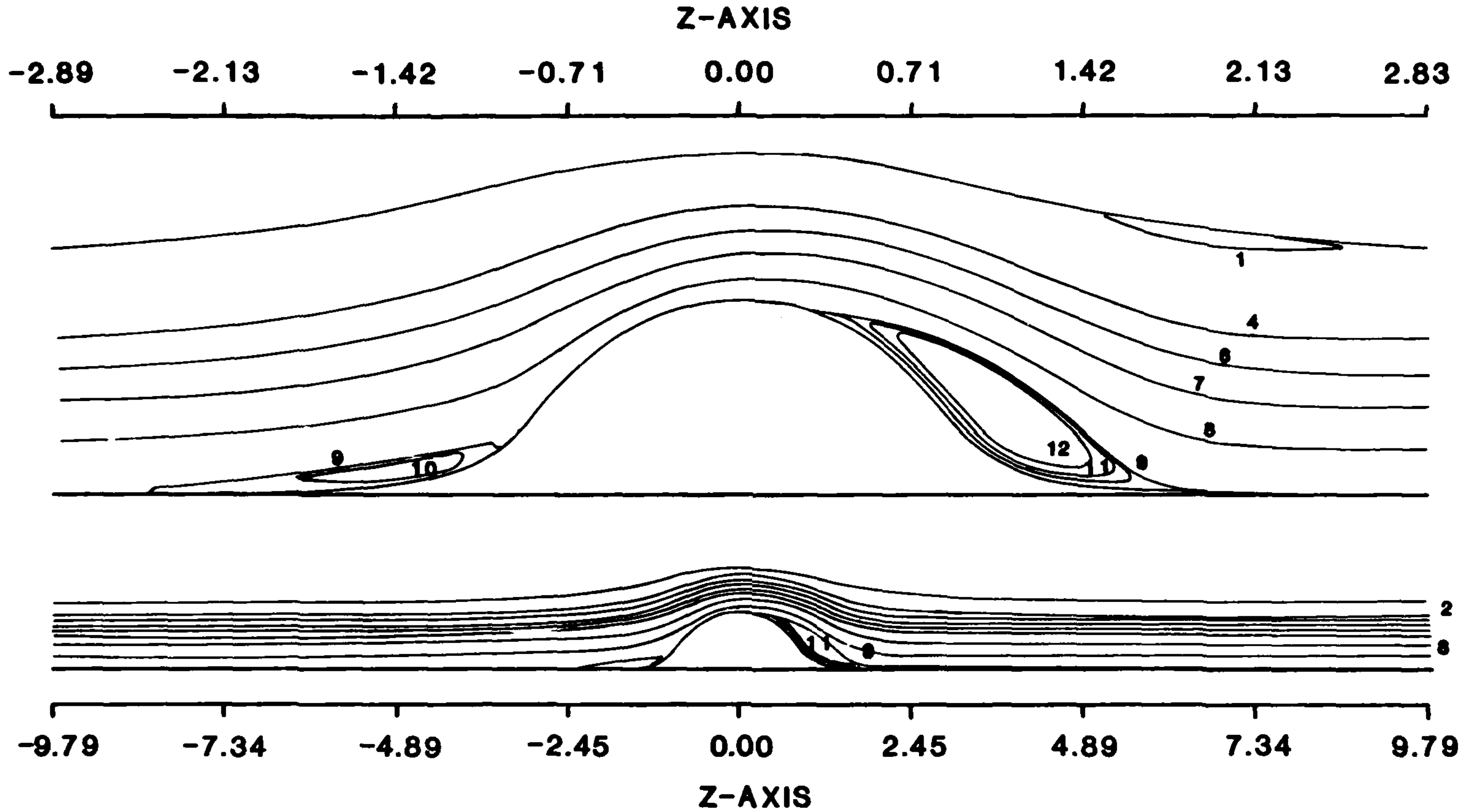

FIGURE 5. STREAMLINE CONTOURS FOR FLOW IN CHANNEL : $R=10^3$ (FROM REF. 14)

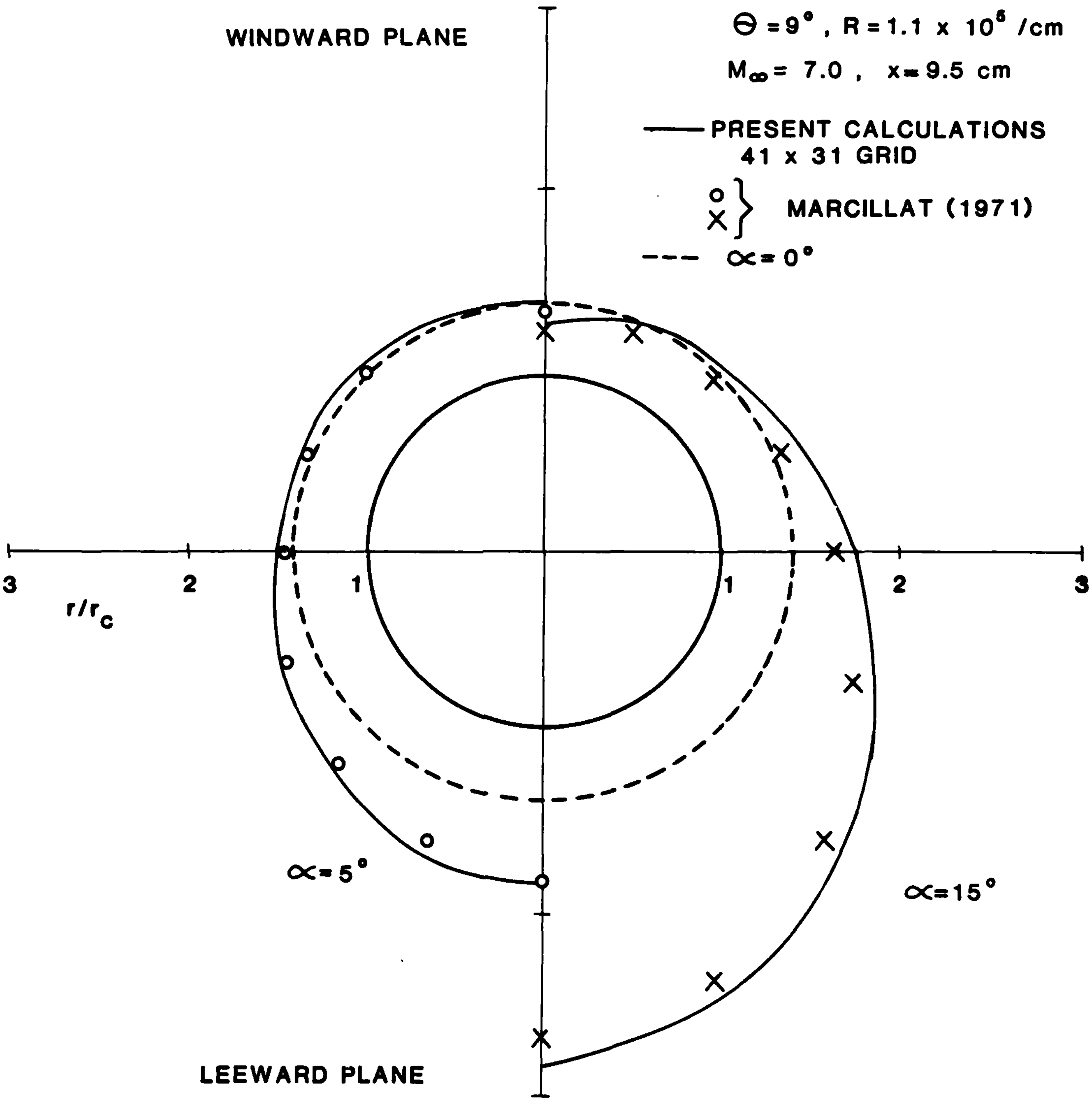

FIGURE 6. SHOCK LOCATION AROUND CONE

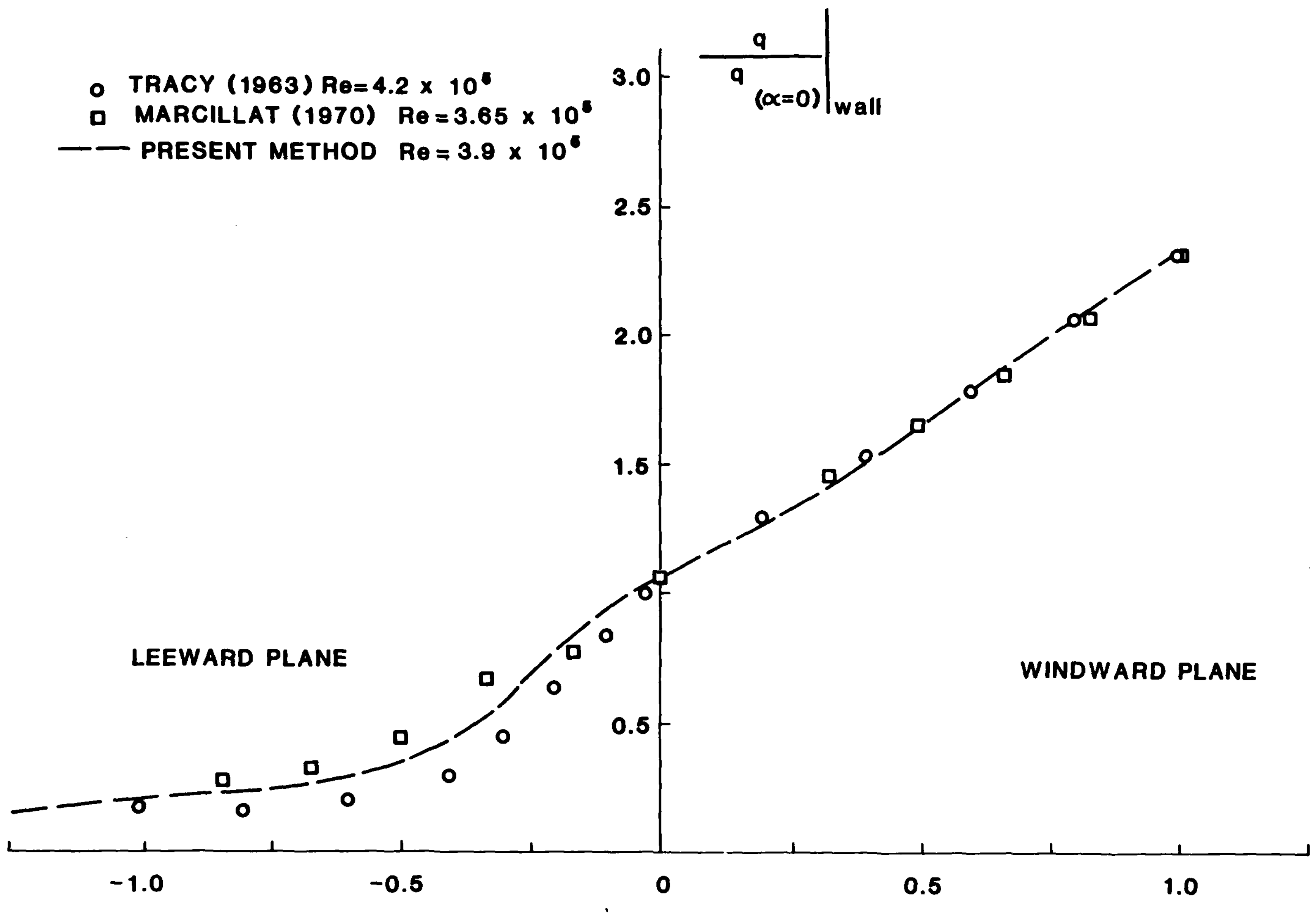

FIGURE 7. SYMMETRY PLANE HEAT TRANSFER

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Marching techniques for the parabolized Navier Stokes equations are considered. With the full pressure interaction and prescribed edge pressure these equations are weakly elliptic in subsonic zones. A minimum marching step size (Δx_{min}), proportional to the total thickness (y_M) of the subsonic layer, exists. However, for thin subsonic boundary layers ($y_M << 1$) and with $\Delta x = O(y_M)$, stable and

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accurate solutions are possible. With forward differencing of the axial pressure gradient the procedure can be made unconditionally stable; a global iteration procedure, requiring only the storage of the pressure term, has been demonstrated for a separated flow problem. Solutions for incompressible boundary layer-like flows, for internal flows, and for supersonic flow over a cone at incidence with a coupled strongly implicit procedure are presented.